RQ:How does the concentration of sucrose in a water solution （0.2M, 0.4M, 0.6M,0.8M,1.0M ± 0.1M） affect the refractive difference between incoming and outgoing angles of light?

Background information

This paper will research the relationship between the concentration of sucrose solution and it's index of refraction. This research will implement the skill of data statistics and experiments on light refraction angles.

Snell's Law is stated in equation 1 below:

$$n_{\mathrm{air}} \times \sin\theta_{\mathrm{air}} = n_{\mathrm{A}} \times \sin\theta_{\mathrm{A}} \quad \text{(Eq.1)}$$

the angle between normal line and income wave θ_{air} as well as the angle between normal line and outgoing wave θ_{A} is related to the refractive index *n*. The index of refraction *n* of the medium *A* is being shown in equation 2 below:

$$n_A = c/v_{\mathrm{A}} \quad \text{(Eq.2)}$$

Where $\mathrm{n_A}$ is the refractive index of the medium *A*; c is the speed of light in vacuum; v_A is the speed of light in the medium *A*.

The Gladstone-Dale relation is shown in equation 3 below:

$$n_{\mathrm{A}} - 1 = k \times \rho_{\mathrm{A}} \quad \text{(Eq.3)}$$

Where *k* is a constant which depends on factors like the material and the wavelength of light, with units of $\mathrm{m^3kg^{-1}}$; ρ_{A} is the density of medium A.

Hence, after substituting Eq. (1) and Eq.(2) into Eq. (3), the relationship between the refractive angle when the light travels through air and sucrose solution is:

$$\frac{\sin\theta_{\mathrm{air}} \times n_{air}}{\sin\theta_{\mathrm{A}}} = k \times \rho_{\mathrm{A}} + 1$$

Equation 4 below is the result :

$$\theta_{\mathrm{A}} = \sin^{-1}\left(\frac{\sin\theta_{\mathrm{air}} \times n_{\mathrm{air}}}{k \times \rho_{\mathrm{A}} + 1}\right) \quad \text{(Eq.4)}$$

Where medium A is the sucrose solution. The lab set up shown in figure 1.

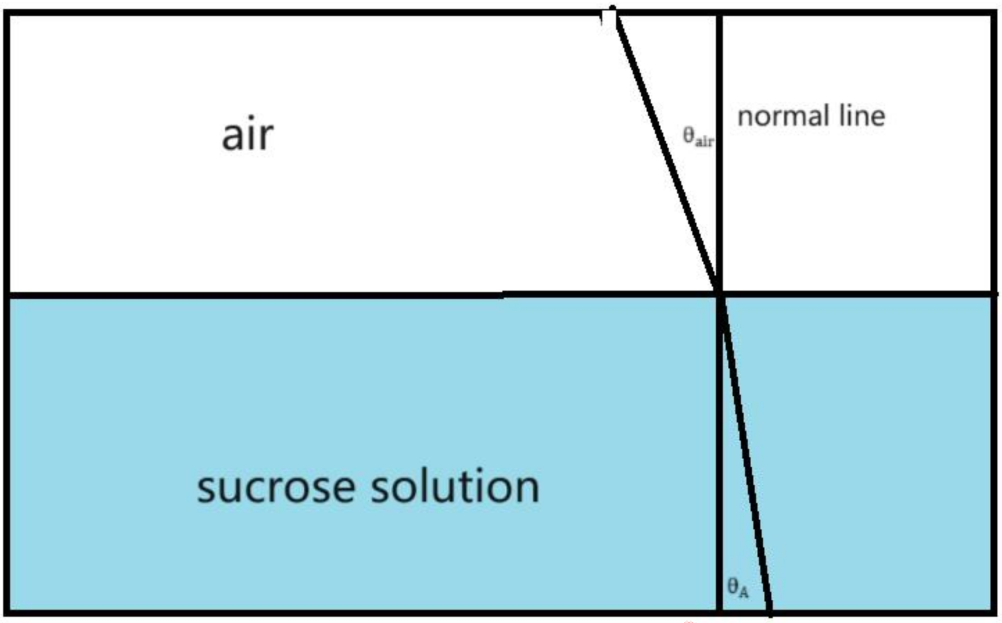


Figure 1. description of the system

In the equation, $\sin\theta_{\text{air}}$ and n_{air} are a controlled variables. During the experiment, ρ_A is the independent variable, causing a change in $\sin\theta_A$. n_{air} is a fixed constant which is 1.0003.

The aim of the investigation and the hypothesis

The investigation's aim is to determine the relationship between the index of refraction and the density difference between two different medias.

From equation 4, it could be hypothesized that the higher the density of the medium, the higher the refraction angle for this medium.This means the smaller angle between the normal and the refracted waves with a constant incoming angle. When ρ_A increase, θ_A decrease. Supports could be found in Anwar ALi's research paper.

The refraction angle will be measured with difference between the light spot on the bottom of the container.

Methodology

experiment variables

Table 1 below list the independent variables and dependent variables in the investigation.

Variables	Details	units
Independent	The concentration (0.2M, 0.4M, 0.6M,0.8M,1.0M)of the sucrose solution in 100cm^3 water. The molarity of solution will being changed to density using the molarity calculator during the calculation.	mol/L (M)
Dependent	The angle between normal line and the outgoing wave if angel of incoming wave is being controlled.	degrees

Table 1. The independent variable and the dependent variable

Table 2 below lists the reasons, method and units of controlled variables in the investigation.

Controlled Variables	Why to control	How to control	units
incoming angle	To control the refraction angle between air and distilled water	Fix the laser pen on the tripod with an constant 45 degrees toward the vertical line (normal line of the refraction)	degrees
The density of glass	To make sure the measurement of the degrees of outgoing waves is comparable. No effect of the density of glass on the observed angle.	Use the same glass water tank in different times of experiment.	kg m^{-3}
The air density	To make sure the ratio between n_A and n_{air} is identical in same experiment.	Conduct the investigation indoor during the same period of time.	kg m^{-3}
The thickness of the glass	To make sure more visual error is caused by different thickness of the glass.	Use the same glass water tank in different times of experiment.	m

Table 2. controlled variables.

In order to complete the lab set up, special methodology strategies will be applied. Because the light of laser pen is invisible, the water tank will have scale on both vertical and horizontal perspectives, so that the end point of the laser appears on the bottom of the tank which will be used to measure the angle using the Pythagorean theorem. Secondly, the experiment will only start when the solution reaches an equilibrium state to prevent error caused by water waves. Finally,solution level in the water tank will be strictly controlled during the experiment to make sure increasing the concentration of the solution caused by increasing the amount of sucrose in the solution be eliminated by Reagent pipette. This can eliminate the error caused by the difference in the solution depth.

Materials

Apparatus

Table 2 below list the apparatus used in this investigation.

Apparatus	Quantity	uncertainty
Small glass water tank with scale	1	±2 mL
Reagent pipette	2	/
glass stirring rod	1	/
Distilled water	/	/
electronic balance	/	±0.01g
Ruler	1	±0.5mm
tripod	1	/
Laser pen	1	/

Table 3. Apparatus list

Experimental procedures

1.Fill the water tank with 1L of water.

2.Set the tripod with the laser pen fully fixed on it. Make sure the laser pen is 45 degrees from vertical toward down.

3.Label the water tank at the point which is vertical to the fixed point of the laser pen.
4.Add 68.5g of sucrose into the solution. Measure the distance L_2 (see figure 2) after the solution is mixed well and the water level is being adjusted using a glass stirring rod and reagent pipette respectively.
5.Repeat step 4 five times with different data.
6.Conduct steps 1 to 5 five times and determine the average value each time.

The experimental set up is being illustrated in Figure 2:

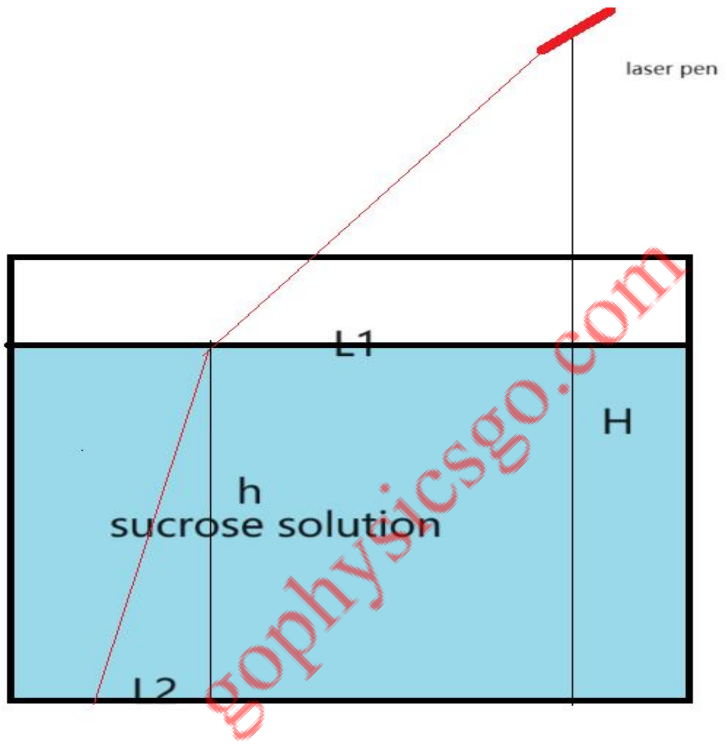


Figure 2. description of the data in experiment set up.

In the actual set up during the experiment, H is the height of the laser source. $H =$ 13cm. h, the height of water surface, is 9 centimeters. L_1 equals 13 centimeters. L_2 ,the difference between the light spot from the position without refraction, is being measured in the experiment with the change in the density of the solution

Risk Assessment

As various rounds the experiment need to collect enough data and reduce random error, water and sucrose will be wasted. During the experiment, around 274 grams of sucrose is used. However, because water and sugar used in the experiment is edible, the solution could be used as cooking ingredients.

Data Analysis

Table 4 below shows the initial data obtained from the experiment:

Solution number	molarity of solution (M)	Solution mass(g)	Δ Solution mass(g)	Solution volume(cm^3)	Δ Solution volume(cm^3)	L_2(cm)	ΔL_2(cm)
1	0	1000.0	0.01	1000	0.5	6.65	0.5
2	0.2	1068.5		1037		5.3	
3	0.4	1137.0		1061		4.8	
4	0.6	1205.5		1098		4.75	
5	0.8	1274.0		1122		4.68	

Table 4. initial raw data

The equation below is used to determine the density of the solution:

$$\rho = \mathrm{m}_{\mathrm{solution}}/\mathrm{V}_{\mathrm{solution}}$$

ρ represents is the density of the solution; m_{solution} is the total mass of the solution measured with an electronic balance and V_{solution} is the volume of the solution. $\mathrm{m}_{\mathrm{solution}}$ is measured by electronic balance. $\mathrm{V}_{\mathrm{solution}}$ is being measured by the the scale on the measuring cylinder. The uncertainty of ρ is $\Delta\rho$. $\Delta\rho$ is being calculated by the following equation:

$$\frac{\Delta\rho}{\rho} = \frac{\Delta m_{\mathrm{solution}}}{\mathrm{m}_{\mathrm{solution}}} + \frac{\Delta V_{\mathrm{solution}}}{V_{\mathrm{solution}}}$$

The uncertainty of the density is calculated as shown below, where x is the solution number:

$$\Delta\rho_{\mathrm{x}} = \rho_{\mathrm{x}}(\frac{\Delta m}{\mathrm{m}_{\mathrm{x}}} + \frac{\Delta V}{V_{\mathrm{x}}})$$

A sample calculation below is made for $\Delta\rho_{\mathrm{x}}$:

$$\Delta\rho_1 = \rho_1(\frac{\Delta m}{\mathrm{m}_1} + \frac{\Delta V}{V_1}) = 1000.0(\frac{1\times 10^{-5}}{1} + \frac{5\times 10^{-7}}{1\times 10^{-3}}) = 0.51 \approx 0.5\mathrm{kg/m^3}$$

In addition, the equation below is used to find the angle θ_A ,between normal line and refracted wave:

$$\theta_{\mathrm{A}} = \tan^{-1}\left(\frac{L_2}{h}\right)$$

The L_2 is the difference of the light spot from the position without refraction: h is the height of the water surface, which is 9 centimeters. A sample calculation below is made for θ_{A1}:

$$\theta_{A\,1} = \tan^{-1}\left(\frac{L_{2\,1}}{h}\right) = \tan^{-1}\left(\frac{0.0665}{0.09}\right) = 36.46° = 36.5°$$

Due to the rule of Propagation of errors (n.d.), the uncertainty of θ_A is $\Delta\theta_A$. $\Delta\theta_A$ is being calculated by the following equation:

$$\Delta\theta_A = \sqrt{\left(\frac{d\theta_A}{dL_2} \times \Delta L_2\right)^2 + \left(\frac{d\theta_A}{dh} \times \Delta h\right)^2}$$

Because of the equation showing the relationship between θ_A and L_2 is

$$\theta_A = \tan^{-1}\left(\frac{L_2}{h}\right)$$

Hence, the partial derivative of F with respect to L_2 is

$$\frac{d\theta_A}{dL_2} = \frac{d\left[\tan^{-1}\left(\frac{L_2}{h}\right)\right]}{dL_2} = \frac{1}{1 + \left(\frac{L_2}{h}\right)^2} \times \frac{1}{h} = \frac{h}{{L_2}^2 + h^2}$$

Similarly, the partial derivative of F with respect to h is

$$\frac{d\theta_A}{dh} = \frac{d\left[\tan^{-1}\left(\frac{L_2}{h}\right)\right]}{dL_2} = \frac{1}{1 + \left(\frac{L_2}{h}\right)^2} \times \left(-\frac{L_2}{h^2}\right) = -\frac{L_2}{{L_2}^2 + h^2}$$

Substitute $\frac{d\theta_A}{dL_2}$ and $\frac{d\theta_A}{dL_2}$ into the equation:

$$\Delta\theta_A = \sqrt{\left(\frac{h}{{L_2}^2 + h^2} \times \Delta L_2\right)^2 + \left(-\frac{L_2}{{L_2}^2 + h^2} \times \Delta h\right)^2}$$

A sample calculation below is made for $\Delta\theta_{A1}$:

$$\Delta\theta_{A\,1} = \sqrt{\left(\frac{h}{{L_{2\,1}}^2 + h^2} \times \Delta L_{2\,1}\right)^2 + \left(-\frac{L_{2\,1}}{{L_{2\,1}}^2 + h^2} \times \Delta h\right)^2}$$

$$= \sqrt{\left(\frac{0.09}{0.0665^2 + 0.09^2} \times 0.005\right)^2 + \left(-\frac{0.0665}{0.0665^2 + 0.09^2} \times 0.005\right)^2}$$

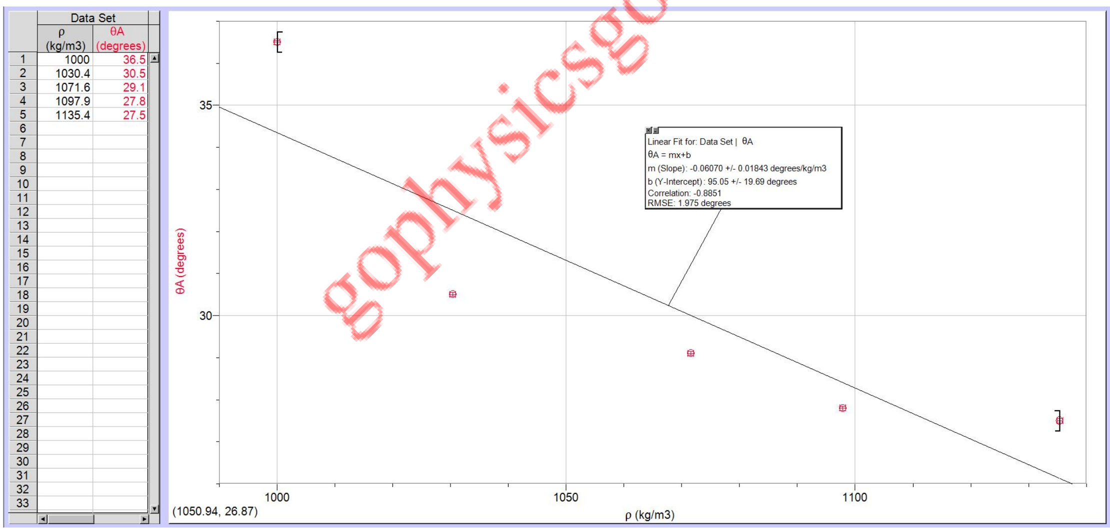
$$= 0.044681° \approx 0.04°$$

The table 5 below list the processed data and their uncertainties:

Solution number	Density of the solution (kg/m^3)		Angle between normal line and outgoing wave (degrees)	
	ρ	$\Delta\rho$	θ_A	$\Delta\theta_A$
1	1000.0	± 0.5	36.5	± 0.04
2	1030.4	± 0.5	30.5	± 0.05
3	1071.6	± 0.5	28.1	± 0.05
4	1097.9	± 0.5	27.8	± 0.05
5	1135.4	± 0.5	27.5	± 0.05

Table 5. density and angle between normal line and outgoing wave

Graph1 illustrates the trend of the relationship between the density of the solution ρ and the angle between normal line and refracted wave θ_A:



Graph 1. Density - angle linear fit

Graph 1 shows that as the density of the solution increases, then the θ_A decreases. In addition, as the density decreases, $\sin\theta_A$ decreases. According to Snell's law, as the density decreases, n_A increases. Nevertheless, graph 1 illustrates the fact that the relationship between the density of the solution and the angle θ_A is probably not linear. The best fit line in the graph had hardly fit into any of the error bars of the collected data. From the graph, θ_A is approximately 34.4 degrees when the refraction occurs is between distilled water and air. As the refractive index of water and air are known (1.0 and 1.3 respectively), with 45 degrees incident angle, θ_A should approximately equals to 33.3 degrees. The uncertainty of θ_A should equal ± 0.5 degrees, which is smaller than the actual difference. This illustrates the systematic error in the experiment. The error could appear because of the difference between tap water and distilled water as well as the mistakes made during set up. Possible mistakes made during the set up could be the laser pen not perfectly 45 degrees from the water surface or the rising in the height of water surface is not considered. Moreover, due to the trend of the data implied in the graph, it is reasonable to suggest that the true relationship between θ_A and $\sin\theta_A$ follows the trend illustrated in equation 4, where as:

$$\theta_A = \sin^{-1}\left(\frac{\sin\theta_{air} \times n_{air}}{k \times \rho_A + 1}\right) \quad \text{(Eq.4)}$$

To find out how the refractive index of the solution is related to the density of the solution, the refractive index is being calculated using equation 1:

$$n_{air} \times sin\,\theta_{air} = n_A \times sin\,\theta_A \quad \text{(Eq.1)}$$

$$\frac{n_{air} \times \sin\theta_{air}}{\sin\theta_A} = n_A \quad \text{(Eq.1)}$$

According to the equation 1, n_A and its corresponding density and θ_A is listed in table 6 below:

Solution number	density of the solution (kg/m^3)	Angle between normal line and outgoing wave (degrees)	Refractive index	
			n_A	Δn_A
1	1000.0	36.5	1.19	± 0.7
2	1030.4	30.5	1.39	± 0.8
3	1071.6	28.1	1.50	± 0.9
4	1097.9	27.8	1.51	± 0.9
5	1135.4	27.5	1.53	± 0.9

Table 6, refractive index and density

To determine the uncertainty of the refractive index of the solution, n_A, the following equation is used.

$$\frac{\Delta n_A}{n_A} = \frac{\Delta n_{air}}{n_{air}} + \frac{\Delta \sin\theta_{air}}{\sin\theta_{air}} + \frac{\Delta \sin\theta_A}{\sin\theta_A}$$

In the equation, Δn_{air} is neglected. So the equation is equals to the following equation:

$$\Delta n_A = \left(\frac{\Delta \sin\theta_{air}}{\sin\theta_{air}} + \frac{\Delta \sin\theta_A}{\sin\theta_A}\right) \times n_A$$

Due to the rule of Propagation of errors (n.d.):

$$\Delta \sin\theta_{\text{air}} = \frac{d[\Delta \sin\theta_{\text{air}}]}{d\theta_{\text{air}}} \times \Delta\theta_{\text{air}} = \cos(\theta_{\text{air}}) \times \Delta\theta_{\text{air}}$$

Similarly,

$$\Delta\, sin\, \theta_A = \frac{d[\Delta\, sin\, \theta_A]}{d\theta_A} \times \Delta\theta_A = \cos(\theta_A) \times \Delta\theta_A$$

Hence, the equation below is used to determine Δn_{A}:

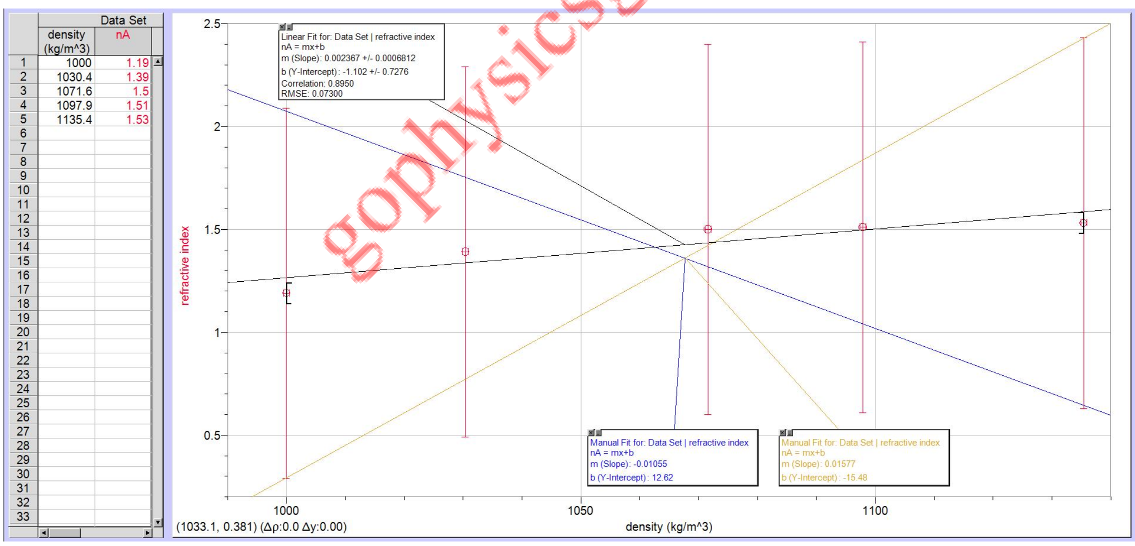
$$\Delta n_{\text{A}} = \left(\frac{\cos(\theta_{\text{air}}) \times \Delta\theta_{\text{air}}}{\sin\theta_{\text{air}}} + \frac{\cos(\theta_{\text{A}}) \times \Delta\theta_{\text{A}}}{\sin\theta_{\text{A}}}\right) \times n_{\text{A}}$$

A sample calculation below is made for Δn_A:

$$\Delta n_{\text{A}} = \left(\frac{\cos(\theta_{\text{air}}) \times \Delta\theta_{\text{air}}}{\sin\theta_{\text{air}}} + \frac{\cos(\theta_{\text{A}}) \times \Delta\theta_{\text{A}}}{\sin\theta_{\text{A}}}\right) \times n_{\text{A}}$$

$$= \left(\frac{\cos(45°) \times 0.5°}{\sin 45°} + \frac{\cos(36.5°) \times 0.04°}{\sin 36.5°}\right) \times 1.19 = 0.659 \approx 0.7$$

Graph 4 below illustrates the relationship between n_{A} and ρ. The relationships between n_{A} and ρ is suggested to be linear.



Graph 2. Density - refractive index linear fit

In Graph 2, the black line with slope 0.002367 is the best fit line. The blue line with slop -0.01055 is the worst fit line with the minimum slope. The yellow line with slope 0.01577 is the worst fit line with the maximum slope. In the graph, a linear relationship could fit in to the actual data collected. However, it is doubtful that the actual relationship between n_{A} and ρ is linear due to the larger uncertainty. Δn_{A} is large showing the random error exists in the experiment. This might be caused by a lack of repetitive experiments to decrease the random error. In addition, the measuring tool also has a low precision, causing high random errors for the raw data.

Suggesting the relationship between the refractive index of solution and the density of the solution is linear, the equation below have illustrates their relationship:

$$n_{\mathrm{A}} = 0.002367\rho + 0.0006812$$

Suggesting m is the slope of the linear fit of the relationship between refractive index of the solution and the density of the solution, its uncertainty is calculated as the equation shown belong:

$$\Delta m = \frac{m_{\max} - \mathrm{m}_{\min}}{2}$$

Substitute $m_{\max}$ and $m_{\min}$ with the slop of worst fit lines, the equation is equals to:

$$\Delta m = \frac{m_{\max} - m_{\min}}{2} = \frac{0.01577 - (-0.01055)}{2} = 0.01316 \approx 0.01\mathrm{m}^3/\mathrm{kg}$$

Conclusion

The hypothesis of the result of the experiment is correct: The relationship between density of the solution and the angle between the normal line and the refracted waves is negative. The refractive index and density of the sucrose solution is positive. The data of the refractive index of solution with various density shows the linear relationship between the refractive index and the density of the sucrose solution, meaning the research question about the relationship between sucrose concentration of the solution and its refractive index is solved. With the research of Anwar ALi as reference it is not difficult to determine the error of my own experiment. In the table 7 below, experimental data from Anwar ALi is listed:

Solution density ($\mathrm{kg/m^3}$) (±)	**Refractive index (±)**				**Average refractive index (±)**
997.56	1.3322	1.3316	1.3314	1.3310	1.3315
1000.1	1.3332	1.3324	1.3323	1.3318	1.3324
1002.7	1.3345	1.3344	1.3333	1.3327	1.3337
1005.1	1.3355	1.3345	1.3344	1.3340	1.3346
1007.4	1.3366	1.3359	1.3351	1.3346	1.3355
1009.8	1.3367	1.3366	1.3357	1.3355	1.3361

Table 7. Density - refractive data as reference(Anwar ALi)

The slope of the best fit linear line of the data is equal to 0.0003888 which is much smaller than the slope from my data (0.002367). In addition, the refractive index of the solution with 1000.1kg/m^3 density is equal to 1.3324, which is closer to the refractive index of water, 1.33, compared to my data collected (1.19). By analyzing both the data from Anwar ALi and I, the change in refractive index is low compared to the change in solution density. However, a positive relationship exists between those two values.

Evaluation

Concluding from the comparation between the data from Anwar ALi and I, the error is relatively significant in my experiment. Another evidence of it is the high percentage error of the refractive index. There are several limitations which may cause the error. Firstly, utilization of tap water instead of distilled water might cause systematic error. Another limitation is the uncertainty of the measuring equipment, as a ruler have high percentage uncertainty.This increases the random error. Finally, the limitation on the available materials and time makes repetition of the experiment difficult. This might lead to lack of precision. There are also some positive sides of my experiment, as the preparation before the experiment is relatively adequate, making the change in data significant. In addition, because the experiment is being finish in continues period of time, environmental factors have relatively low changes.

Source of error	**Effect on data and errors**	**Improvements**
Displacement in position of light spot	Physical rulre is used to measure, meaning high uncertainty	Use vernier caliper to measure the location.
Wavelength of the laser pen	Might cause difference in the angle	Using wavelength controlled laser pen.
Sugar mass	Sugar mass might be lost during the process of making solution, causing errors in the concentration of the solution	Using pre-existing sucrose solution, skip the process of making the solution

Table 8. Errors and whys to improve

Bibliography

Density, viscosity, and refractive index of mono-, di-, and tri-saccharides in aqueous glycine solutions at different temperatures. Arabian Journal of Chemistry. Anwar ALi. https://www.sciencedirect.com/science/article/pii/S1878535214002007?via%3Dihub

Propagation of errors—basic rules. (n.d.). https://courses.washington.edu/phys431/propagation_errors_UCh.pdf