

**What Is the Relationship between an Object, its Orbital Center, as-well-as
the Nature of the Kepler Constant k in Planetary Motion?**

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1.1.1 Introduction

A few years ago, I enjoyed one of the most impressive movies I had ever watched, *Interstellar*, directed by Christopher Nolan. In the movie, Cooper and his crews were attempting to travel to a mysterious wormhole adjacent to Jupiter in order to reach another solar system that was said to have suitable living condition for the mankind. Therefore, to reach the wormhole, the protagonists implemented a type of space-traveling method called a *gravitational sling-shot* in which the spacecraft utilizes the force of gravity to accelerate itself so that large amounts of fuels will be reserved. After watching the movie, I soon found my curiosity in the relationship between a planet and the smaller celestial bodies surrounding it. As a matter of fact, the sling-shot effect reminded me of the countless meteorites that had hit our planet's ground. Hence, I decided to investigate the relationship between a planet and a smaller object that is in motion and to explore the nature of Kepler constant.

Since the 16th century, Western philosophers and scientists started to fancy the doom hanging above our heads. However, unlike ancient thinkers that established their theories of the space based on spiritual interpretations and creative imaginations, scholars in this period started to conduct investigations dwelling on the foundation of sophisticated experiments and calculations. Thus, in 1618, Johannes Kepler developed his three laws of planetary motion.

1.1.2 Kepler's Laws of Planetary Motion

Kepler's first law of planetary motion implied that all the planets are orbiting around the Sun on their own elliptical routes, and the Sun had to be located on one of the focal points of the ellipse as shown in *Figure 1*; the second law demonstrated that for a given time period, the line that connect the Sun and the planet would sweep across a fixed area, regardless the location of the planet on the orbit as seen in *Figure 1*. Hence, $S_{AB} = S_{CD} = S_{EK}$.

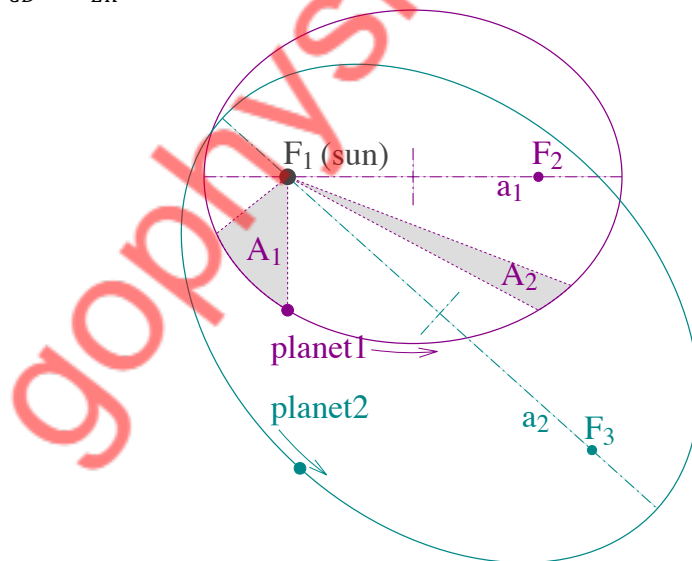


Figure 1. Kepler's first & second law of planetary motion.

For this investigation, I will mostly focus on exploring Kepler's third law of planetary motion, and further determine the nature of the Kepler Constant k . Nevertheless, before starting the exploration, definitions of the physical quantities involved shall be stated to clarify the following steps and hypothesis.

As planets orbit around the Sun following elliptical orbits, their routes can be plotted:

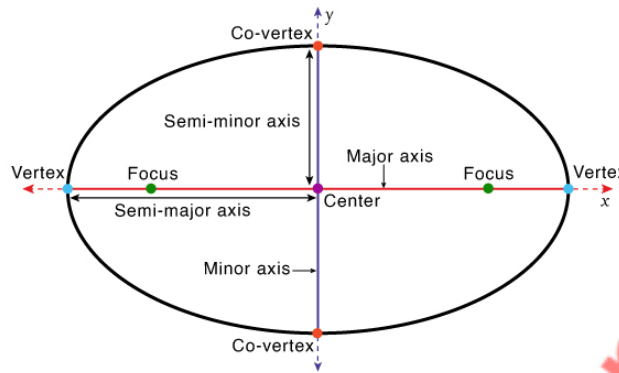


Figure 2. Properties of an Ellipse

In Figure 2, an elliptical orbit has two foci, a major axis and a minor axis. I will make the assumption supported by Kepler's first law of planetary motion that the center of revolution, which is usually a sun or a planet, should be located at one of the foci in its orbit. Also, half the length of the axis is called semi-major axis and semi-minor axis. However, throughout this investigation, I will only be using semi-major axis, denoted by a . Moreover, there are some quantities involved in the investigation. The period of revolution will be denoted as T . Also, the gravitational constant G will be assumed to be $6.667 \times 10^{-11} Nm^2kg^{-2}$.

Kepler's third law of motion discusses the relationship between a planet and the object revolving around it, and the relationship between the physical quantities in its own orbit. Hence, my research question is presented below.

1.2 Research Question

What Is the Relationship between an Object, its Orbital Center, as-well-as the Nature of the Kepler Constant k in Planetary Motion?

1.3 Hypothesis

The cube of the length of the semi-major axis is directly proportional to the square of the period of revolution, the ratio k only depends on the mass of the orbital center, calculated by $\frac{GM}{4\pi^2}$.

2.1 Variable Table

Table 1 below shows the variables appear in this paper.

VARIABLES	DEFINITIONS
a	Length of semi-major axis
k	Kepler constant
G	Gravitational constant
T	Period of revolution
M	Mass of the center
l	Distance from center
v	Initial Velocity

2.2.1 Materials

The following materials will be used to conduct this IA

- A hula-hoop with diameter of 94 cm
- A sheet of Latex, large enough to cover the opening of the hula-hoop
- Iron ball with mass of 1390 g
- Steel weights of 50g, 100g and 200g (see *Figure 4*)
- Marbles
- Stop Watch, with uncertainty of 0.01s
- Laptop's camera
- Black Ink (see *Figure 3*)
- Tape measure with uncertainty of 0.05cm (see *Figure 6*)
- Vernier caliper with uncertainty of 0.05mm (see *Figure 5*)
- Handmade wooden slide

Note: initial force was adjusted by discharging the marbles from prelabelled height on the wooden slide.



Figure 3. Black Ink. Figure 4. Steel Weights. Figure 5. Vernier Caliper. Figure 6. Tapeline.

2.2.2 Ethical and Safety Concerns

The iron balls and the steel weights used in this investigation are heavy and rigid, so be cautious with the items to prevent them from falling on people's feet.

2.3.1 Relationship between Initial Velocity and Period of Revolution

The independent variable is initial velocity of the marbles, achieved by changing the height in which the marbles were released. The dependent variable is the period of revolution, measured by stop watch.

The controlled variables are:

1. Mass of the center object

Reason: Changes in the mass of the center object will change the slope of the depression, thereby may affect the results of the experiment.

Method: Maintain the mass of the center object as 1390 g throughout the experiment.

2. Distance from the center

Reason: Changes in the distance from the center would affect the length of semi-major axis of the orbit, which might influence the measurement.

Method: Placing the slide at a fixed distance away from the center horizontally.

2.3.2 Procedure of Collecting Data

1. Stretch the Latex on the hula-hoop.
2. Place the steel ball with mass of 1390g on the sheet.
3. Dip the marble in black ink to make sure it can leave a clear trace on the sheet.
4. Use slide to discharge the marble at a fixed distance from the center.
5. At the same time, start the stop watch.
6. Pause the stop watch when the marble completes its first revolution.
7. Record the data and repeat steps 2-6 four more times.
8. Change the applied force to low/medium/high using the slide. Repeat steps 2-8 five times.

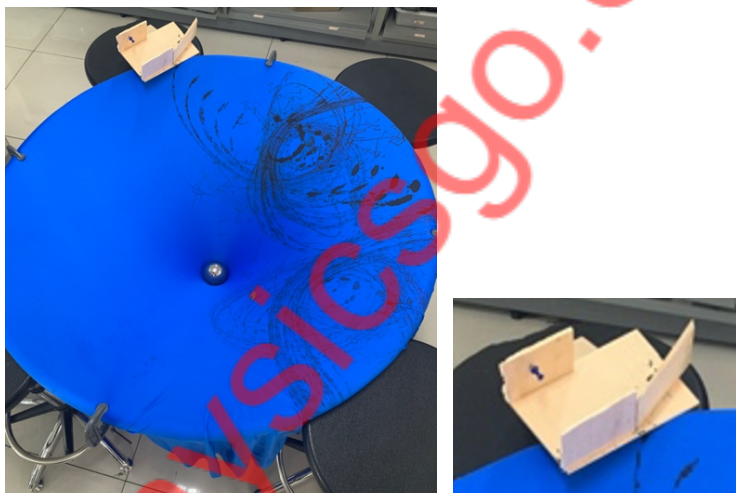


Figure 7. Left: Setup for the experiment. Right: Zoom-in image of the slide.

2.3.3 Results/Raw Data

Table 2 shows the relationship between period and initial speed. Column 1 shows the initial speed of the marble. Column 3 shows the period of revolution. Rows 2-4 tell us the speed of the marbles released.

Initial Speed	Trials	Period of Revolution ($\pm 0.01s$)
High	1	1.56
	2	1.50
	3	1.56
Medium	4	1.47
	5	1.47
	6	1.50
Low	7	1.50
	8	1.46
	9	1.47

The table above shows part of the raw data (see Appendix I). It indicated a brief relationship between the marbles' initial speed and period of revolution. However, when taking a close look into these values, we might find it interesting that there wasn't an apparent correlation between the two variables. In order to test if these variables were statistically insignificant, hence proving the absence of correlation, I decided to implement the *p-test* by using a calculator online.

As the *P-value* occurred to be 0.1011, which is larger than the significance level of 0.05, this difference is considered to be statistically not significant. Hence, we may conclude that there is no cause-effect relationship between the initial speed and period of revolution.

2.4.1 Proving Kepler's Third Law of Planetary Motion

The independent variable is the length of the orbit's semi-major axis, it was achieved by placing the slide at different distances from the center object (20cm, 25cm, 30cm, 35cm, 40cm). The dependent variable is the period of revolution, recorded and measured by Logger Pro.

The controlled variables are:

1. Mass of the center object (see 2.3.1)
2. Initial speed

Reason: Though it is proven that, theoretically, initial speed of the marble displays no role in affecting period of revolution, it would be more secure to eliminate the possible errors, brought by changing initial velocities, that reduce the precision of the results.

Method: Release marbles at the medium height labelled on the slide.

2.4.2 Procedure of Collecting Data

1. Same as step 1-3 in the previous experiment
4. Use the slide to discharge the marble at a radius 20cm from the center
5. Same as step 5-7 in the previous experiment
8. Change the distance to 25cm, 30cm, 35cm, 40cm. Repeat steps 2-7

2.4.3 Establishing Coordinate Surface to Determine the Length of Semi-major Axis

As previously explained, the path taken by planets appear to be elliptical. Thus, there has to be a semi-major axis in the orbit. Nevertheless, the ellipses left by the marbles with black ink were only a minute part of the whole orbit that it supposed to be in vacuum space. Therefore, I decided to measure the length of the semi-major axis by establishing a coordinate system using PowerPoint.

Firstly, after completing all the trials for the experiment, place the latex on a flat surface. Then, measure the distance between the center object and one end of the orbit's major axis by implementing a vernier caliper with uncertainty of $\pm 0.05\text{mm}$. Eventually, export the image to PowerPoint, analyze it by placing a full ellipse on the image, and calculate the length of the semi-major axis implementing geometry.

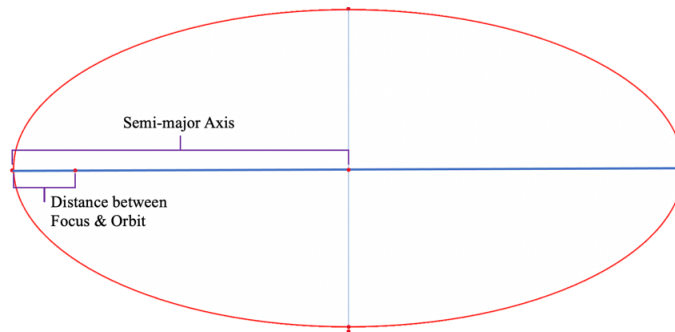


Figure 8. Elliptical Orbit on PowerPoint; Labelled by the Candidate.

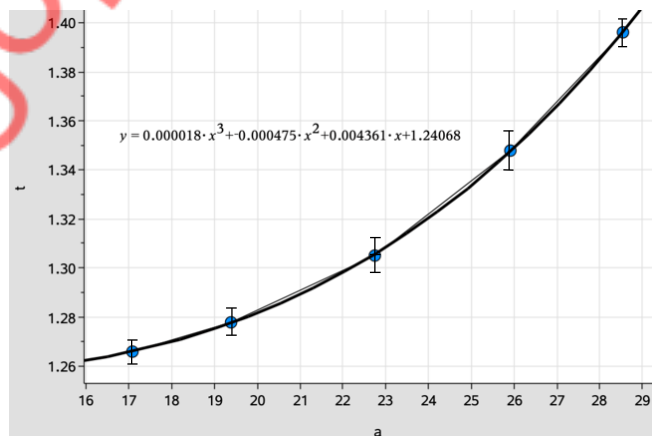
On PowerPoint, once I determined the most suitable ellipse for each image, I would be able to construct a coordinate surface that reflected the relationship between the distance I measured, which was the distance between the focus and the orbit, according to Kepler’s first law, and the semi-major axis of the actual orbit of the marble. Consequently, as long as I know the actual distance measured and could acquire the ratio of it to the length of semi-major axis on PowerPoint, I am capable of calculating the actual length of the semi-major axis.

2.4.4 Results/Raw Data

Table 3 below shows the relationship between the length of the semi-major axis a and the period of revolution T . Column 1 shows the distance from the orbital center (iron ball). Column 2 shows the length of semi-major axis. Column 3 shows the average period of revolution (see Appendix II). Rows 2-6 demonstrate the actual distance from the orbital center (iron ball).

Distance from Center ($\pm 0.5cm$)	Length of Semi-major Axis ($\pm 0.005cm$)	Av. Period of Revolution ($\pm 0.002s$)
20.0	17.090	1.266
25.0	19.380	1.278
30.0	22.740	1.305
35.0	25.920	1.348
40.0	28.540	1.396

Graph 1 below shows the change in period of revolution (T) to the length of semi-major axis (a).

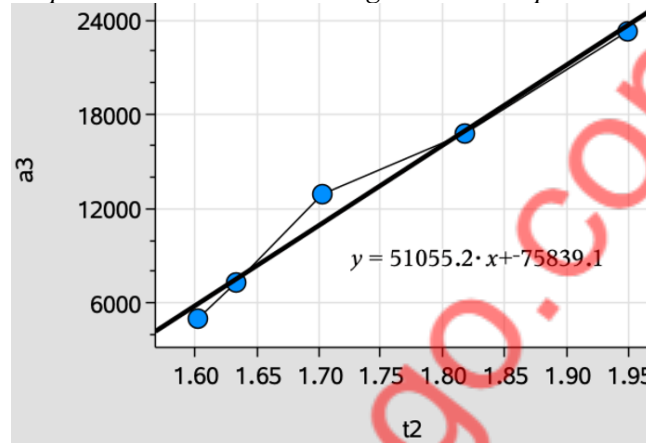


By simply illustrating the diagram of the relationship between T and a , I found that these two variables are not directly proportional to each other. However, after fitting a regression line, I found that T and a appeared as the following cubic relationship:

$$y = 0.000018 \cdot x^3 - 0.000475 \cdot x^2 + 0.004361 \cdot x + 1.24068$$

Thus, a diagram representing the Change in T^2 in respect to a^3 was laid out (see Appendix II).

Graph 2 below shows the change in a^3 in respect to T^2 .



Referencing to *Graph 2*, I can draw the conclusion that the cube of the length of semi-major axis is directly proportional to the square of the period of revolution. This finding corresponds to the hypothesis previously made; hence the ratio will be denoted as k in the rest of the paper. However, it is not guaranteed that this curve is accurately obeying Kepler's original graph and calculation because:

1. The latex I used and the space in which the experiment was conducted are yielding resistance on the marble. Therefore, it is a systematic error that adds inaccuracy to the raw data.
2. The slope created by the concaved latex made it harder for me to launch the marble; in fact, at 35cm and 40cm from the center, the marble disposed actually bounced at the beginning of the rotation. Therefore, it is a random error that reduces precision of the raw data.

2.5.1 Literature Review, Graph Analysis and Calculation to Determine the Expression of k .

According to *Figure 8* and NASA, there is a perihelion (closest distance between the Sun and the planet) and an aphelion (farthest distance between the Sun and the planet) in a planetary orbit. Thus, the mean distance of perihelion and aphelion is going to be the length of semi-major axis for a planet in the solar system. With reference to the *Planetary Fact Sheet* presented by NASA (see Appendix III), I was able to lay out some possible correlations in a real-world context:

Table 4. The Value of k for Planets in the Solar System. (Excluded Neptune & Pluto).

Planet	Distance from Sun (10^6 km)	Orbital Period (days)	a^3/T^2 (km^3/day^2)
Mercury	57.9	88.0	25.0652
Venus	108.2	224.7	25.0886
Earth	149.6	365.2	25.1034
Mars	228.0	687.0	25.1126
Jupiter	778.5	4331.0	25.1536
Saturn	1432.0	10747.0	25.4246
Uranus	2867.0	30589.0	25.1856

Graph 3 below shows the change in T^2 in respect to a^3 in the solar system.

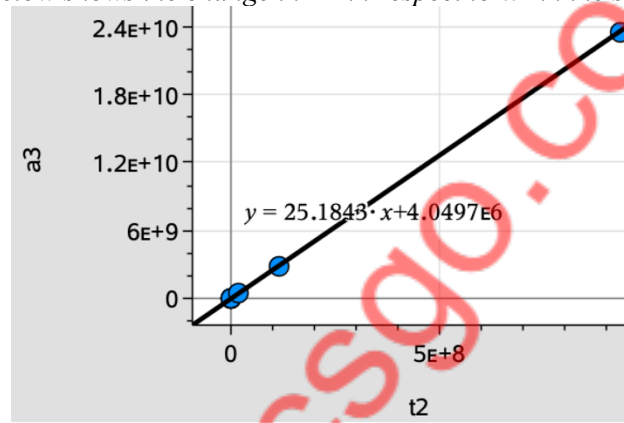


Figure 13 and the third column of Table 3 suggest the idea that the major planets in the solar system obey the law of planetary motion proved in the previous section. However, as NASA also presented the data for Moon (which is rotating around the Earth), there tends to be an interesting phenomenon. Since the center of rotation for Moon is the Earth, the perihelion and aphelion were measured as the distance from Earth. Therefore, the value of k for Moon is calculated as below:

$$k_{moon} = \frac{a^3}{T^2} = \frac{0.384^3}{27.3^2} = 0.0007579 \text{ km}^3/\text{day}^2$$

Comparing to the average value of k for planets orbiting around the Sun, the Moon tends to have a significantly lower k . This discovery leads me to fancy if the constant k might be affected by the mass of the Sun in which an object is orbiting around, meaning that the reason for Moon to have such a tiny k is because it is simply orbiting around the Earth rather than the Sun itself.

Originally, I was thinking of conducting the third experiment by varying the mass of the center object on the latex. Nevertheless, as in the school's lab I could only make minute changes, if any, on the mass of the center object, the results would demonstrate a mere correlation that k varies in respect to mass. Thus, I decided to use mathematical calculation, combined with Kepler's First & Second Law of planetary motion, to prove my hypothesis. The following steps are conducted:

If the perihelion and aphelion of an orbit are denoted as A and B, with C denoting the center object, then \mathbf{v}_a and \mathbf{v}_b represent the instantaneous speeds that tangent to the elliptical orbit. Hence, the areal velocity at A and B are calculated by:

$$\mathbf{S}_A = \frac{1}{2}(\mathbf{r}_a \times \mathbf{v}_a) = \frac{1}{2}(a - c) \times \mathbf{v}_a$$

$$\mathbf{S}_B = \frac{1}{2}(\mathbf{r}_b \times \mathbf{v}_b) = \frac{1}{2}(a + c) \times \mathbf{v}_b$$

Note: a is the length of semi-major axis, c is the distance between center object and orbital center. S_A and S_B are the areal velocity of A and B separately.

As stated by Kepler's Second Law of planetary motion (see section 1.1.2), $\mathbf{S}_A = \mathbf{S}_B$, so by establishing an equation, \mathbf{v}_b is calculated as:

$$\mathbf{v}_b = \frac{a - c}{a + c} \cdot \mathbf{v}_a$$

The mechanical energy E of a planet is the sum of its total kinetic and potential energy. Given that $E_A = E_B$ due to the law of conservation of energy, we have:

$$E_A = \frac{1}{2}mv_a^2 - \frac{GMm}{a - c} = E_B = \frac{1}{2}mv_b^2 - \frac{GMm}{a + c}$$

Hence,

$$\frac{1}{2}m(v_a^2 - v_b^2) = GMm\left(\frac{1}{a - c} - \frac{1}{a + c}\right)$$

By replacing \mathbf{v}_b with \mathbf{v}_a , we have:

$$\mathbf{v}_a = \sqrt{\frac{(a + c)GM}{a(a - c)}}$$

Therefore,

$$\mathbf{S}_A = \mathbf{S}_B = \mathbf{S} = \frac{b}{2} \sqrt{\frac{GM}{a}}$$

Note: b is the length of semi-minor axis, it is calculated by $\sqrt{a^2 - c^2}$.

Since the area of an ellipse is πab , the period of revolution is therefore:

$$T = \frac{\pi ab}{S} = 2\pi a \sqrt{\frac{a}{GM}}$$

After squaring both sides,

$$\frac{a^3}{T^2} = \frac{GM}{4\pi^2} = k$$

Therefore, it is proven mathematically that the value of k only depends on the mass M of the center.

4.1 Conclusion

To sum up with, after conducting two experiments along with a theoretical calculation process regarding Kepler's law of planetary motion, the two stated hypothesis are mostly answered as it was proven by *Graph 2* that, during planetary motion, the square of the period of revolution is directly proportional to the cube of the length of semi-major axis. Combining the results from the first experiment, *Graph 3*, and the calculation process after that, I will also be able to generate the thesis that the value of the ratio (denoted k) is changing IFF (if and only if) the mass of the center object, in the case of the solar system, the Sun, is changing. Therefore, I believe that, despite some occurrence of inaccuracy and imprecision during the experiments, the investigation as a whole has supported my hypothesis to a considerable extent. Nevertheless, in order to reach perfection in certain parts during the exploration, there are still some issues which could be adjusted so that a more satisfying outcome would come through.

4.2 Errors and Limitations of the Investigation

As long as we know the exact expression of Kepler constant and the mass of the center object, we would be able to determine the exact value of k . Therefore, the slope resulted from experiment 1, which should equal to k , could be compared with the calculated actual value as below:

$$k = \frac{GM}{4\pi^2} = \frac{6.667 \times 10^{-11} \text{Nm}^2\text{kg}^{-2} \times 1.390\text{kg}}{4\pi^2} \approx 2.347 \times 10^{-12} \text{m}^3/\text{s}^2$$

In comparison, the slope in *Graph 2* is 51055.2. However, this value has a unit of cm^3/s^2 , so the slope becomes $5.106 \times 10^{-2} \text{m}^3/\text{s}^2$. The huge difference between the actual value and the measured value indicates that some significant limitations and errors that affected the precision and accuracy of the raw data were occurring in my experiment.

First of all, in contrast with Kepler's observation of planetary motion in space, the experiments I conducted allowed the presence of resistance. In space, planets are strictly obeying Kepler's law of planetary motion due to the absence of air resistance. However, the marble in my experiment was experiencing resistive force from both the air and latex. This, as a matter of fact, resulted in increase in systematic errors, thereby boost the inaccuracy of the raw data. Furthermore, the marble was even bouncing on the latex once it was launched in some trials of the experiment, in which doubtlessly added uncertainties to the data collected, and provided a significantly inaccurate regression line in *Graph 2*.

Furthermore, during the experiment, I made the assumption that when the marble passed through the line that connects the center object and the location where it was disposed (shown in *Figure 9*), it completed half of the full rotation period, hence the distance between the intersection point and the initial location would be measured as the orbit's major axis.

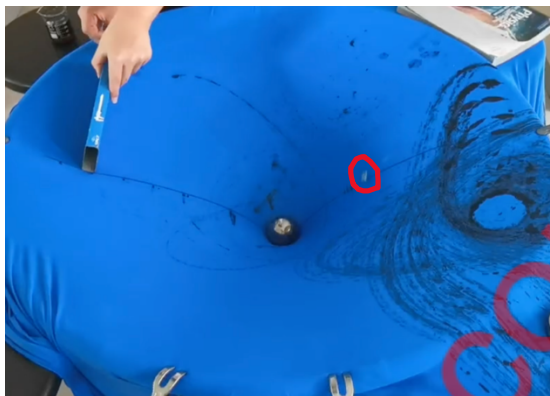


Figure 9. Demonstration of the Intersection Point.

Nevertheless, in practical planetary motion, the Earth is revolving around the Sun without falling to it because the Sun itself is also moving—due to attraction from its own *Sun* in the middle of the galaxy—in front of the Earth. Therefore, the Earth is able to sustain a stable pattern of rotation which has its identical period and length of semi-major axis. Nonetheless, in the experiment the marble was guaranteed to descend into the center object. This slight difference may result in distortion of the original orbit of the marble as it could not maintain on its stable pattern of revolution. Thus, the assumption previously made may not be accurate, and due to the inclusion of cubic value in the expression of k , this slight inaccuracy could cause tremendous changes in the final result.

Lastly, the limitation of the experiments was due to the minute mass of the iron ball in comparison with the mass of a planet, the expression of k could only be proven theoretically using Kepler's first and second laws of planetary motion instead of directly proven by the raw data collected since the changes in k with respect to the changes in mass of the center object was too insignificant to be observed and measured. Therefore, the last part of the investigation was completely based on literature review and hypothetical analysis using mathematical calculations. Unfortunately, this was inevitable because it is impossible to establish an object that has the mass significant enough for experimenters to determine the relationship between mass and the value of k .

4.3 Suggested Methods for Improvement

Due to the nature of the Kepler's law of planetary motion, there are not many effective adjustments for us to make in a school lab in order to acquire a perfect final result. However, there are still some plausible suggestions that will increase the precision of the data collected.

In experiment 2, I only conducted one trial for each distance from the center for the sake of convenient collection process and clarity of the data because with the marble dipped into ink, the trace left on the latex would get tangled together if excessive trials were conducted. However, this led to the decrease in precision of the final data since it was guaranteed that random errors would occur during the collection

process. Therefore, I think that with the help of Photoshop or some video-editing software, it will be possible for me to abandon the use of ink during the experiments as such software are able to follow the marble automatically so that an outline of the trace will be plotted. This enables me to conduct each trial more than once in order to reduce random errors, hence enhance the precision of the data, especially when *Graph 2* illustrated that the third datapoint was obviously distant from the regression line.

However, there are few, if any, possible adjustments I could conduct to make the experiments more accurate. Therefore, even though the results did provide me with a brief insight of the connections involved within an elliptical orbit, they failed to match the actual results generated from proper, accurate experiments. The graceful truth that dwells in between the stars is probably only available for us when we observe it relentlessly with curiosity, just like what Kepler did a few centuries ago.

5.1 Bibliography:

1. First, second and third editions © K.A.Tsokos 1998, 1999, 2001 Fourth, fifth, fifth (full colour) and sixth editions © Cambridge University Press 2005, 2008, 2010, 2014
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4. Johannes Kepler, *Harmonices Mundi* [The Harmony of the World] (Linz, (Austria): Johann Planck, 1619), book 5, chapter 3, p. 189.
5. <https://www.cuemath.com/geometry/ellipse/>

Appendix I

Period of Revolution (*measured by stopwatch*) under different Initial Velocities

Initial Velocity	Trials	Period of Revolution ($\pm 0.01s$)
High	1	1.56
	2	1.5
	3	1.56
	4	1.54
	5	1.53
Medium	6	1.47
	7	1.55
	8	1.5
	9	1.53
	10	1.53
Low	11	1.5
	12	1.46
	13	1.57
	14	1.47
	15	1.52

Appendix II

Distance from the center (cm)	Period of revolution ($\pm 0.002s$)	Length of semi-major axis ($\pm 0.005cm$)	Distance from the center ($\pm 0.005s$)
20	1.266	17.090	14.460
25	1.278	19.380	13.790
30	1.305	22.740	15.530
35	1.348	25.920	17.400
40	1.396	28.540	16.320

Appendix III

Planetary Fact Sheet.

Source: <https://nssdc.gsfc.nasa.gov/planetary/factsheet/index.html>

	MERCURY	VENUS	EARTH	MOON	MARS	JUPITER	SATURN	URANUS	NEPTUNE	PLUTO
Mass (10 ²⁴ kg)	0.330	4.87	5.97	0.073	0.642	1898	568	86.8	102	0.0130
Diameter (km)	4879	12,104	12,756	3475	6792	142,984	120,536	51,118	49,528	2376
Density (kg/m ³)	5429	5243	5514	3340	3934	1326	687	1270	1638	1850
Gravity (m/s ²)	3.7	8.9	9.8	1.6	3.7	23.1	9.0	8.7	11.0	0.7
Escape Velocity (km/s)	4.3	10.4	11.2	2.4	5.0	59.5	35.5	21.3	23.5	1.3
Rotation Period (hours)	1407.6	-5832.5	23.9	655.7	24.6	9.9	10.7	-17.2	16.1	-153.3
Length of Day (hours)	4222.6	2802.0	24.0	708.7	24.7	9.9	10.7	17.2	16.1	153.3
Distance from Sun (10 ⁶ km)	57.9	108.2	149.6	0.384*	228.0	778.5	1432.0	2867.0	4515.0	5906.4
Perihelion (10 ⁶ km)	46.0	107.5	147.1	0.363*	206.7	740.6	1357.6	2732.7	4471.1	4436.8
Aphelion (10 ⁶ km)	69.8	108.9	152.1	0.406*	249.3	816.4	1506.5	3001.4	4558.9	7375.9
Orbital Period (days)	88.0	224.7	365.2	27.3*	687.0	4331	10,747	30,589	59,800	90,560
Orbital Velocity (km/s)	47.4	35.0	29.8	1.0*	24.1	13.1	9.7	6.8	5.4	4.7
Orbital Inclination (degrees)	7.0	3.4	0.0	5.1	1.8	1.3	2.5	0.8	1.8	17.2
Orbital Eccentricity	0.206	0.007	0.017	0.055	0.094	0.049	0.052	0.047	0.010	0.244
Obliquity to Orbit (degrees)	0.034	177.4	23.4	6.7	25.2	3.1	26.7	97.8	28.3	122.5
Mean Temperature (C)	167	464	15	-20	-65	-110	-140	-195	-200	-225
Surface Pressure (bars)	0	92	1	0	0.01	Unknown*	Unknown*	Unknown*	Unknown*	0.00001
Number of Moons	0	0	1	0	2	79	82	27	14	5
Ring System?	No	No	No	No	No	Yes	Yes	Yes	Yes	No
Global Magnetic Field?	Yes	No	Yes	No	No	Yes	Yes	Yes	Yes	Unknown
	MERCURY	VENUS	EARTH	MOON	MARS	JUPITER	SATURN	URANUS	NEPTUNE	PLUTO